Convective Instability in Packed Beds with Throughflow

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The purpose of this note is to present the results of numerical calculations of linear stability limits for free convection in layers of porous media or packed beds with throughflow. Effects of flow direction and of different boundary conditions are shown. Previous results were given by Homsy and Sherwood (1976) for the special boundary conditions of constant velocity and temperature at upper and lower boundaries. These results were valid for any flow rate and direction. Wooding (1960) calculated linear stability limits for a semiinfinite layer with large upward flow when the upper surface was porous and submerged by a layer of liquid at constant temperature. Sutton (1970) presented results valid for small flow rates. For packed-bed applications, e.g., the reaction zone in a catalytic reactor or an ion exchange column, the porous or, as it is sometimes called, constant-pressure boundary condition is more appropriate. In this work, we show results for all combinations of boundary conditions and flow direction. The theory and results of calculations are presented within the framework of thermal convection. As is well known, complete analogy exists with concentration-driven convection and we only have to replace T with concentration, thermal diffusivity α' with the species diffusivity, and the thermal expansivity β with the appropriate linear concentration coefficient of density.

The linear theory has been given previously, for example by Homsy and Sherwood, and by Wooding. In summary, marginal states were calculated from Darcy's law and the conduction-convection energy equation. Property variations (density and viscosity) were assumed to be small—the Boussinesq approximation—and linear with temperature from some reference temperature T_0:

$$\rho = \rho_o[1 - \beta(T - T_o)], \quad \beta(T - T_o) \ll 1 \tag{1}$$

$$\mu = \mu_o[1 + \beta'(T - T_o)], \quad \beta'(T - T_o) \ll 1 \tag{2}$$

where, for gases, both β and β' are positive. After elimination of the pressure and restatement in dimensionless form with the height H of the layer as the reference length, the resulting eigenvalue problem for the mass flux perturbation Γ and the temperature perturbation θ is obtained by the usual normal mode analysis:

$$(D^2 - a^2)\Gamma = -\lambda a^2\theta \tag{3}$$

$$[Pe'D - (D^2 - a^2)]\theta = -F(z)\Gamma \tag{4}$$

In these equations, $D = d/dz$ and z is the dimensionless vertical coordinate with origin at the lower extremity of the porous layer. The parameter $Pe' = HG/\alpha'\rho_o$ is a modified thermal Peclet number in which $\alpha' = k_m/\rho C_{pf}$ is the modified thermal diffusivity and G is the steady mass flux, which is positive for upward flow. The parameter λ is given by $\lambda = g\beta H\Delta TK/\nu_o\alpha' - (\beta + \beta')Pe'\Delta T$. The first term in this expression is the Rayleigh-Darcy number and the second term is a contribution due to the throughflow. $a = \underline{k}H$ is the dimensionless wave number of the disturbance in the horizontal plane. $F(z)$ is the dimensionless steady state temperature gradient.

A single fourth-order ordinary differential equation is obtained by eliminating θ in favor of Γ:

$$[-Pe'D(D^2 - a^2) + (D^2 - a^2)^2 + \lambda a^2F(z)]\Gamma = 0, \tag{5}$$

with appropriate boundary conditions at $z = 0$ and 1. For constant mass flux (impermeable to perturbations) and temperature, $\Gamma = 0, \theta = 0$, or, in terms of Γ only,

$$\Gamma = 0, \quad \Gamma'' = 0 \tag{6}$$

For constant-temperature boundaries that are porous or permeable to flow perturbations $\Gamma' = 0, \theta = 0$, or

$$\Gamma' = 0, \quad \Gamma'' = a^2\Gamma \tag{7}$$

There are thus four possible combinations of these boundary conditions, which may be denoted as follows for upper and lower boundaries, respectively:

Impermeable/impermeable
Porous/impermeable
Impermeable/porous
Porous/porous.

In addition, there are two possible throughflow directions for upward and downward flow, which are manifested in the function $F(z)$ and the sign of G. However, the eigenvalue problems for the two flow directions are identical when the boundary conditions are identical at upper and lower boundaries. Furthermore, porous/impermeable boundary conditions with upward flow give the same eigenvalue problem as impermeable/porous with downward flow, so in fact only four distinct eigenvalue problems are possible for any given value of Pe'.

In the present work, the eigenvalues λa^2 were determined by expressing Eq. 5 in finite-difference form with second-order approximation for all derivatives with respect to z. For a chosen grid, this leads to a matrix eigenvalue problem for which the eigenvalues were determined by using the Fortran routine EIGRF (IMSL, 1982). The smallest eigenvalue is then an *approximation* to the critical value of λa^2 for a given a. The critical values of both λ and a were determined by minimizing with respect to a. Grids of 20, 50, and 100 subdivisions were tested and it was found that sufficient accuracy could be obtained with 50, with λ differing by less than 0.1% from the grid of 100 subdivisions.

The quantity λ can be written as $Ra - (\beta + \beta')Pe'\Delta T$, where Ra is the conventional Rayleigh-Darcy number for Bénard convection. The second term can be related to flow instability caused by the displacing fluid expanding (speeding up) in the direction of flow or having lower viscosity, as discussed by Homsy and Sherwood and by Wooding. This term contributes to instability in downward flow, $Pe' < 0$, but has a stabilizing effect in upward flow. Anticipating the results of the calculations, we find that λ/Pe' is always greater than 1. But by the Boussinesq approximation, $(\beta + \beta')\Delta T$ is always to be much less than 1. Therefore, the second term in λ may not be dominant within the framework of the theory. This is not to say that flow instability, say, under low gravity or low permeability conditions may not be dominant, but that the present theory is then inapplicable. Thus in the present calculations, we can to a good approximation identify λ with the Rayleigh-Darcy number Ra.

The resulting critical Ra values are shown in Figure 1 plotted against Pe' for the case

$$F(z) = \frac{-Pe'e^{Pe'z}}{e^{Pe'} - 1}. \tag{8}$$

i.e., the steady state convection-conduction (diffusion) temperature (concentration) profile. In this figure, we have plotted only results for positive Pe', i.e., upward flow, but it should be noted that the impermeable/impermeable results are symmetrical about $Pe' = 0$. The same is true for the porous/porous results. The results for the porous/impermeable case for negative Pe' are identical to those for impermeable/porous for positive Pe' and vice versa. Thus complete results may be obtained from this figure. The following points are noted:

1. Porous boundary conditions give lower critical Ra than impermeable, the lowest being when both are porous.
2. For high flow rates, the results depend only on the boundary condition for the boundary toward which the flow is directed.
3. Linear relationships for Ra vs. Pe' are obtained for large Pe' in all cases, as previously observed by Homsy and Sherwood.
4. For zero Pe', the well-known results are recovered, namely, $Ra = 4\pi^2$, 27.1, and 12.0 for impermeable, mixed, and porous boundary conditions, respectively. This was a required test of the calculations. (See Nield, 1968 Table 1, and 1984.)
5. Except for a shallow minimum at $Pe' = 1$ for flow toward a porous boundary, throughflow is always stabilizing. The origin of the minimum is not completely understood, but we have eliminated numerical inaccuracy as a possible cause.

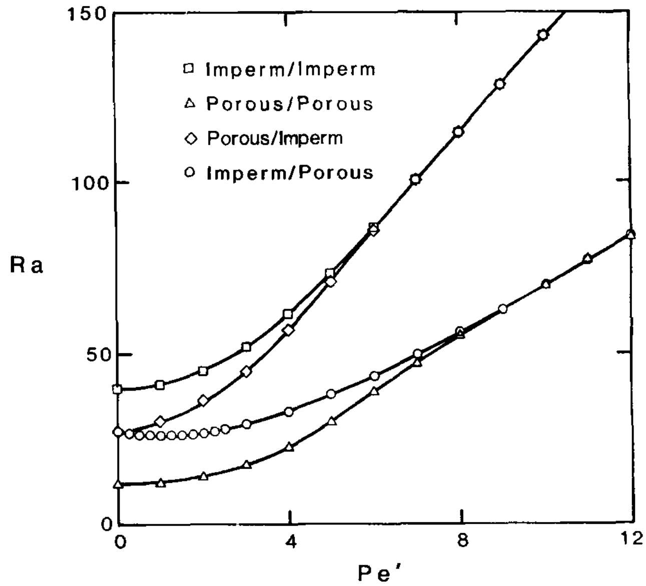

Figure 1. Linear stability limits for combinations of upper/lower boundary conditions using steady state temperature gradient of Eq. 8.

With reference to points 2 and 5, these results are undoubtedly due to the strong steady state temperature gradient $F(z)$ being confined to an increasingly thinner layer as Pe' is increased. The free convection must then be of a boundary layer nature and become independent of the boundary at which flow enters. The problem then should be independent of the height of the layer H, and the problem should be rescaled. Choosing as reference length $\rho_o\alpha'/G$, we find the resulting eigenvalue problem now contains only one characteristic parameter $\lambda' = \lambda/Pe'$. Thus,

$$[D(D^2 - a^2) + (D^2 - a^2)^2 + \lambda' a^2 F(z)]\Gamma = 0, \tag{9}$$

where $F(z) = -\exp(-z)$. The boundary conditions were specified as the porous or impermeable type according to Eqs. 6 or 7 at the boundary toward which the flow was directed, and at an arbitrary boundary layer thickness δ we required

$$\Gamma' = 0, \quad \theta = 0 \tag{10}$$

The parameter λ' was determined as in the previous calculations for values of δ. When δ was increased, λ' reached a stationary

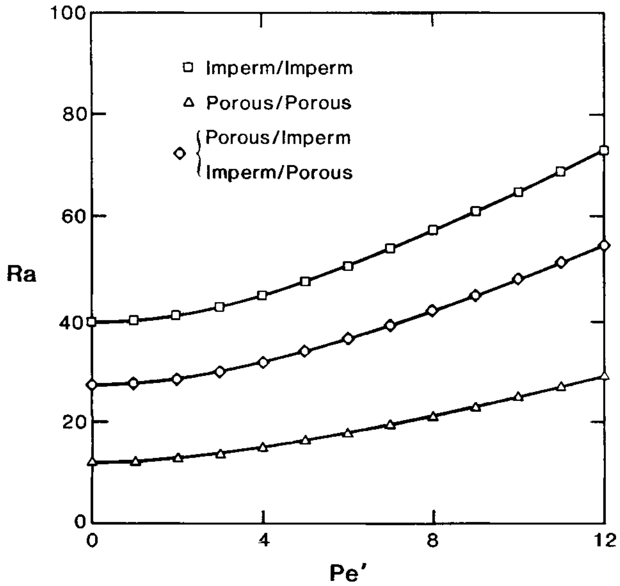

Figure 2. As for Figure 1, but with linear steady state temperature.

value at about $\delta = 8$. The asymptotic values thus obtained for λ' were 14.2 and 7.05 for impermeable and porous boundaries, respectively. These numbers are in agreement, as they should be, with the slopes of the asymptotes of Figure 1, and are within the accuracy of calculation for the values reported by Homsy and Sherwood for an impermeable boundary and by Wooding for a porous boundary.

The steady state temperature gradient profile given by Eq. 8, particularly in its boundary layer form, may not be a very realistic representation of a chemical reactor, where the profile may be governed by the evolution of heat. We therefore repeated the calculations of Figure 1 with a simple linear temperature profile, i.e., $F = -1$. This situation may also represent a transient unperturbed state due to the high heat capacity of the solid. The results are given in Figure 2; they show that throughflow is still stabilizing, but now to a lesser extent. The asymmetry noted in Figure 1 for the two mixed boundary situations no longer exists and no boundary layer behavior is observed. Such stabilization as occurs must now be understood in terms of the convection of disturbances out of the system by the throughflow.

Notation

a = wave number = Hk
C_{pf} = heat capacity of fluid, J/kg · K
D = differential operator, = d/dz
F = unperturbed temperature gradient
g = gravitational acceleration, m/s^2
G = steady-state vertical mass flux, kg/m^2 · s
H = height of layer, m
K = permeability, m^2
k = wave number, m^{-1}
$\bar{k}_m$ = effective conductivity of medium, W/m · K
T = temperature
ΔT = temperature difference across bed, K
z = dimensionless vertical coordinate

Greek letters

α' = modified thermal diffusivity, m^2/s
β = thermal expansivity, K^{-1}
β' = temperature coefficient of viscosity, K^{-1}
Γ = mass flux perturbation
δ = thermal boundary layer thickness
λ = stability parameter defined in text
μ = viscosity, kg/m · s
ν = kinematic viscosity, m^2/s
θ = temperature perturbation
ρ = density, kg/m^3

Subscripts

o = value at reference temperature T_o

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———, Personal communication (1984). (The last line of Table 1. in Nield (1968) was in error. For porous/porous boundary conditions, the critical Rayleigh-Darcy number with zero throughflow is 12.0, not zero.)
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